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Kinetics of the long-range spherical model

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Abstract
The kinetic spherical model with long-range interactions is studied after a quench to $T < T_c$ or to $T = T_c$. For the two-time response and correlation functions of the order parameter as well as for composite fields such as the energy density, the ageing exponents and the corresponding scaling functions are derived. The results are compared to the predictions which follow from local scale invariance.

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1. Introduction

A many-body system rapidly brought out of some initial state by quenching it to either a critical point or into a coexistence region of the phase diagram where there are at least two equivalent equilibrium states undergoes *ageing* [1]. For ageing systems, the physical state evolves slowly, non-exponentially and depends on the time since the quench was performed and hence time-translation invariance is broken. In addition, there holds some kind of dynamical scaling, whether or not the stationary states are critical. These aspects of ageing can be conveniently studied through the two-time response and correlation functions defined as

$$R(t,s) = \left.\frac{\delta\langle\mathcal{O}(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r})}\right|_{h=0} = s^{-a-1} f_R\left(\frac{t}{s}\right), \qquad f_R(y) \overset{y\to\infty}{\sim} y^{-\lambda_R/z}, \tag{1.1}$$

$$C(t,s) = \langle\mathcal{O}(t,\boldsymbol{r})\mathcal{O}(s,\boldsymbol{r})\rangle = s^{-b} f_C\left(\frac{t}{s}\right), \qquad f_C(y) \overset{y\to\infty}{\sim} y^{-\lambda_C/z}, \tag{1.2}$$

where the observable $\mathcal{O}(t,\boldsymbol{r})$ (at time t and location $\boldsymbol{r}$) is typically taken to be the order-parameter $\phi(t,\boldsymbol{r})$. In this work, we shall also study composite fields such as the energy

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density. We denote by h the field conjugate to $\mathcal{O}$ (and when $\mathcal{O}$ is the order parameter the conjugate field h is the associated magnetic field). The dynamical scaling forms (1.1), (1.2) are expected to hold in the scaling limit where both $t, s \gg t_{\text{micro}}$ and also $t-s \gg t_{\text{micro}}$, where t_{micro} is some microscopic time scale. In writing equations (1.1), (1.2), it is implicitly assumed that the underlying dynamics is such that there is a single relevant length-scale $L = L(t) \sim t^{1/z}$, where z is the dynamical exponent (the presence of another relevant large length scale would break dynamical scaling). Non-equilibrium universality classes are distinguished by different values of exponents such as $a, b, \lambda_C, \lambda_R$ (which will depend on the observable $\mathcal{O}$ and the field h used and also on whether $T < T_c$ or $T = T_c$). For reviews, see [1–5].

In trying to find a systematic approach to determine the scaling functions $f_{R,C}$ it has been proposed to generalize the dynamical scaling to a *local* scale invariance [6, 7] which include the transformation $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$ in time with $\alpha\delta - \beta\gamma = 1$. In a field-theoretical setting [8, 9], the autoresponse function can be formally rewritten as a correlator $R(t, s) = \langle \phi(t)\widetilde{\phi}(s)\rangle$ where $\widetilde{\phi}$ is the response field associated with ϕ. From the assumption that both ϕ and $\widetilde{\phi}$ are so-called *quasi-primary* scaling operators [7, 10] (see section 3), it follows that

$$f_R(y) = f_0 y^{1+a'-\lambda_R/z}(y-1)^{-1-a'}\Theta(y-1), \tag{1.3}$$

where $\Theta(y)$ is the Heaviside function which expresses causality, a' is an exponent and f_0 a normalization constant. A similar explicit, if lengthy, expression can be given for the autocorrelation. We refer to [11–13] for recent reviews on the derivation of these results and on the numerous examples where these predictions have been tested. For our purposes, it is enough to note that most of these tests are done in situations where the dynamical exponent $z = 2$. In particular, almost all existing tests for $z \neq 2$ merely tested the prediction (1.3), and this for the order parameter *only*, see [11, 12] for a detailed discussion. The only exception are a few simple models where $z = 4$ [14, 15].

A fuller picture on the validity of the several technical assumptions which are needed for the precise formulation of the theory of local scale invariance (LSI) can only come from more systematic tests of its predictions. To this end, we shall study in this paper the ageing behaviour of the spherical model with long-range interactions. It was shown by Cannas, Stariolo and Tamarit [16] that for quenches to $T < T_c$ if the exchange couplings decay sufficiently slowly with the distance then the dynamical exponent z becomes a continuous function of the control parameters of the model and that the scaling forms (1.1), (1.2) hold for the order parameter. Here, we shall extend these considerations to the critical case $T = T_c$ and shall further look at the scaling behaviour of composite operators (i.e. energy density). Specifically, we shall enquire

(i) whether dynamical scaling holds, and if so, what are the values of the corresponding non-equilibrium exponents?
(ii) what is the form of the scaling functions of responses and correlators?
(iii) which of the composite operators, if any, transform as quasi-primary fields under local scale invariance?

In section 2, we review the exact solution of the kinetic long-range spherical model and list our results for the non-equilibrium exponents and the scaling functions for the order parameter and for composite fields. Some of the details are treated in the appendix. In section 3, we first show that the presently available formulation [7] of local scale invariance cannot explain our results on the spacetime form of the response functions when $z \neq 2$. We then announce some results of a forthcoming paper [17] on a general reformulation of local scale invariance for $z \neq 2$ before comparing our explicit results with the corresponding predictions of that general theory. In section 4 we conclude.

2. Exact solution of the long-range spherical model

The two-time correlation and response functions of the order parameter in the spherical model when quenched either to $T = T_c$ or else to $T < T_c$ are well-known in the case of nearest-neighbour interactions [18–22]. These are also known for the long-range model when quenched to $T < T_c$ [16]. Here, we shall derive the response and correlation functions of the order parameter and of certain composite operators in the long-range mean spherical model quenched to $T \leqslant T_c$.

2.1. Long-range spherical model

The long-range spherical model is defined in terms of a real spin variable $S(t, \boldsymbol{x})$ at time t and on the sites $\boldsymbol{x}$ of a d-dimensional hypercubic lattice $\Lambda \subset \mathbb{Z}^d$, subject to the (mean) spherical constraint

$$\left\langle \sum_{\boldsymbol{x}\in\Lambda} S(t, \boldsymbol{x})^2 \right\rangle = \mathcal{N}, \tag{2.1}$$

where $\mathcal{N}$ is the number of sites of the lattice[5]. The Hamiltonian is given by [24]

$$\mathcal{H} = -\frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{y}} J(\boldsymbol{x}-\boldsymbol{y}) S_{\boldsymbol{x}}(S_{\boldsymbol{y}} - S_{\boldsymbol{x}}), \tag{2.2}$$

where the sum extends over all pairs $(\boldsymbol{x}, \boldsymbol{y})$ such that $\boldsymbol{x} - \boldsymbol{y} \neq \boldsymbol{0}$. The coupling constant $J(\boldsymbol{x})$ of the model is defined by

$$J(\boldsymbol{x}) = \left({\sum_{\boldsymbol{y}\in\Lambda}}' |\boldsymbol{y}|^{-(d+\sigma)} \right)^{-1} |\boldsymbol{x}|^{-(d+\sigma)}, \tag{2.3}$$

when $\boldsymbol{x} \neq 0$ and vanishes when $\boldsymbol{x} = 0$; the summation is over all lattice sites except $\boldsymbol{y} = 0$. The last term in (2.2), $\sum_{\boldsymbol{x},\boldsymbol{y}} J(\boldsymbol{x}-\boldsymbol{y}) S_{\boldsymbol{x}}^2$, can also be absorbed into the Lagrange multiplier that imposes the spherical constraint, see below.

The 'usual' spherical model with short-range interactions is given by $J_{sr}(\boldsymbol{x}-\boldsymbol{y}) = J\sum_{\boldsymbol{\mu}(\boldsymbol{x})} \delta_{\boldsymbol{y},\boldsymbol{x}+\boldsymbol{\mu}(\boldsymbol{x})}$, where $\boldsymbol{x}+\boldsymbol{\mu}(\boldsymbol{x})$ runs over all the neighbouring sites of $\boldsymbol{x}$. When $\sigma \geqslant 2$, the relevant large-scale behaviour of the above model, (2.2) and (2.3), is governed by this short-range model. Here, we shall focus on truly long-range interactions such that $0 < \sigma < 2$. In this case, the dynamical exponent $z = \sigma$ can be continuously varied by tuning this parameter, see [16, 24] and below. For the equilibrium behaviour of the model, consult the classic review by Joyce [24].

The dynamics is governed by the Langevin equation[6]

$$\partial_t S(t, \boldsymbol{x}) = -\left.\frac{\delta\mathcal{H}}{\delta S_{\boldsymbol{x}}}\right|_{S_{\boldsymbol{x}}\to S(t,\boldsymbol{x})} - \mathfrak{z}(t) S(t, \boldsymbol{x}) + \eta(t, \boldsymbol{x}), \tag{2.4}$$

where the coupling to the heat bath at temperature T is described by a Gaussian noise η of vanishing average and a variance

$$\langle \eta(t, \boldsymbol{x})\eta(t', \boldsymbol{x}')\rangle = 2T\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{2.5}$$

The Lagrange multiplier $\mathfrak{z}(t)$ is fixed by the mean spherical constraint.

[5] For short-ranged interactions, a careful analysis [23] has shown that the long-time behaviour is not affected whether (2.1) is assumed exactly or on average.

[6] In equation (2.4), fluctuations in the Lagrange multiplier $\mathfrak{z}(t)$ are neglected. As pointed out in [22], these must be taken into account when treating non-local observables involving spins from the entire lattice or if the initial magnetization is nonzero. Here, we are only interested in local quantities and use a vanishing initial magnetization. See [25] for a careful discussion on the applicability of Langevin equations in long-ranged systems.

The Langevin equation and the variance of the noise in the Fourier space read

$$\partial_t \widehat{S}(t, \boldsymbol{k}) = -(\omega(\boldsymbol{k}) + \mathfrak{z}(t))\widehat{S}(t, \boldsymbol{k}) + \widehat{\eta}(t, \boldsymbol{k}), \tag{2.6}$$

$$\langle \widehat{\eta}(t, \boldsymbol{k})\widehat{\eta}(t', \boldsymbol{k}')\rangle = 2T(2\pi)^d \delta(t - t')\delta(\boldsymbol{k} + \boldsymbol{k}'), \tag{2.7}$$

where $\omega(\boldsymbol{k}) = \widehat{J}(\boldsymbol{0}) - \widehat{J}(\boldsymbol{k})$. The hatted functions denote the Fourier transform of the corresponding functions. In the long-wavelength limit $|\boldsymbol{k}| \to 0$, the function $\omega(\boldsymbol{k}) \to B|\boldsymbol{k}|^\sigma$, where the constant B is given by [16] $B = \lim_{|\boldsymbol{k}|\to 0}(\widehat{J}(\boldsymbol{0}) - \widehat{J}(\boldsymbol{k}))|\boldsymbol{k}|^{-\sigma}$.

The solution of the above equation is

$$\widehat{S}(t, \boldsymbol{k}) = \frac{\mathrm{e}^{-\omega(\boldsymbol{k})t}}{\sqrt{g(t; T)}} \left[\widehat{S}(0, \boldsymbol{k}) + \int_0^t \mathrm{d}\tau\, \mathrm{e}^{\omega(\boldsymbol{k})\tau} \sqrt{g(\tau; T)}\widehat{\eta}(\tau, \boldsymbol{k})\right], \tag{2.8}$$

with the constraint function $g(t; T) = \exp\left(2\int_0^t \mathrm{d}\tau \mathfrak{z}(\tau)\right)$. The system is assumed to be quenched from far above the critical temperature, hence $\langle \widehat{S}(0, \boldsymbol{k})\rangle = 0$; and the spins are assumed to be uncorrelated initially, hence the spherical constraint implies $\langle \widehat{S}(0, \boldsymbol{k})\widehat{S}(0, \boldsymbol{k}')\rangle = (2\pi)^d \delta(\boldsymbol{k} + \boldsymbol{k}')$. Therefore, the spin–spin correlation function when $t > s$ is

$$\langle \widehat{S}(t, \boldsymbol{k})\widehat{S}(s, \boldsymbol{k}')\rangle = (2\pi)^d \delta(\boldsymbol{k} + \boldsymbol{k}')\widehat{C}(t, s; \boldsymbol{k}), \tag{2.9}$$

where

$$\widehat{C}(t, s; \boldsymbol{k}) = \frac{\mathrm{e}^{-\omega(\boldsymbol{k})(t+s)}}{\sqrt{g(t; T)g(s; T)}} \left[1 + 2T \int_0^s \mathrm{d}\tau\, \mathrm{e}^{2\omega(\boldsymbol{k})\tau} g(\tau; T)\right]. \tag{2.10}$$

The spherical constraint implies $1 = \int_{\Lambda_k} \widehat{C}(t, t; \boldsymbol{k})$ and gives $g(t; T)$ as the solution to the Volterra integral equation [16, 20]

$$g(t; T) = f(t) + 2T \int_0^t \mathrm{d}\tau f(t - \tau)g(\tau; T), \tag{2.11}$$

with $g(0; T) = 1$, and $f(t) = f(t, \boldsymbol{0})$ is obtained from the function

$$f(t; \boldsymbol{r}) := \int_{\Lambda_k} \mathrm{d}\boldsymbol{k}\, \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - 2\omega(\boldsymbol{k})t), \tag{2.12}$$

where $\Lambda_{\boldsymbol{k}}$ denotes the first Brillouin zone of the lattice Λ.

2.2. *Composite operators: correlations and responses*

We shall now consider not only the spin operator $S(t, \boldsymbol{r})$ but also some composite fields, specifically the spin-squared (spin^2) operator and the energy-density operator. We denote the spin and spin^2 operators by

$$\mathcal{O}_1(t, \boldsymbol{x}) := S(t, \boldsymbol{x}), \tag{2.13}$$

$$\mathcal{O}_2(t, \boldsymbol{x}) := S^2(t, \boldsymbol{x}) - \langle S^2(t, \boldsymbol{x})\rangle, \tag{2.14}$$

respectively. The energy-density operator is defined as

$$\begin{aligned} \mathcal{O}_\epsilon(t, \boldsymbol{x}) &:= \mathcal{E}(t, \boldsymbol{x}) - \langle \mathcal{E}(t, \boldsymbol{x})\rangle, \\ \mathcal{E}(t, \boldsymbol{x}) &:= \sum_{\boldsymbol{x}'} J(\boldsymbol{x} - \boldsymbol{x}')S(t, \boldsymbol{x})(S(t, \boldsymbol{x}') - S(t, \boldsymbol{x})). \end{aligned} \tag{2.15}$$

These composite operators are defined in such a way that their average value is zero, and hence their correlation functions are essentially the *connected* correlation functions. Also note that since the energy is defined only up to a constant there is no unique definition of the energy-density operator.

The distinction between $\mathcal{O}_2$ and $\mathcal{O}_\epsilon$ might be better understood as follows. We look into the continuum limit of the energy-density operator, at least for short-range model, for we shall later discuss that this operator is not quasi-primary under local scale-invariance. In the short-range model, the expression for energy in lattice models is usually taken as

$$\mathcal{H} = -J \sum_{\boldsymbol{x},\boldsymbol{\mu}(\boldsymbol{x})} S_{\boldsymbol{x}} S_{\boldsymbol{x}+\boldsymbol{\mu}(\boldsymbol{x})}, \tag{2.16}$$

where $\boldsymbol{x} + \boldsymbol{\mu}(\boldsymbol{x})$ runs over the neighbouring sites of $\boldsymbol{x}$. In such a case, the energy density could be defined as $\widetilde{\epsilon}(\boldsymbol{x}) = -J \sum_\mu S_{\boldsymbol{x}} S_{\boldsymbol{x}+\boldsymbol{\mu}}$, which in the continuum limit would reduce to $\widetilde{\epsilon}(\boldsymbol{x}) = -J\left(2S_{\boldsymbol{x}}^2 + \mu^2 S_{\boldsymbol{x}} \nabla^2 S_{\boldsymbol{x}}\right)$, where μ is the lattice constant. But if we had added an overall constant $E_0 = \mathcal{N} = \sum_x S_x^2$ then the energy density could be defined as

$$\epsilon(\boldsymbol{x}) = -J \sum_{\boldsymbol{\mu}} S_{\boldsymbol{x}}(S_{\boldsymbol{x}+\boldsymbol{\mu}} - S_{\boldsymbol{x}}) \to -J\mu^2 S_{\boldsymbol{x}} \nabla^2 S_{\boldsymbol{x}}(1 + \mathrm{O}(\mu)). \tag{2.17}$$

Hence $\mathcal{H}_{\mathrm{sr}} = \sum_{\boldsymbol{x}} \epsilon(\boldsymbol{x}) = J\mu^2 \sum_{\boldsymbol{x}} (\nabla S_{\boldsymbol{x}})^2$, up to boundary terms. Therefore, for our model (2.2) the two operators $\mathcal{O}_2(t, \boldsymbol{x})$ and $\mathcal{O}_\epsilon(t, \boldsymbol{x})$ must be distinguished.

The connected two-point correlation functions of the composite operators

$$\mathcal{C}_{ab}(t, s; \boldsymbol{x} - \boldsymbol{x}') := \langle \mathcal{O}_a(t; \boldsymbol{x}) \mathcal{O}_b(s; \boldsymbol{x}') \rangle \tag{2.18}$$

are obtained by making use of Wick's contraction as detailed in the appendix. Throughout it is implicitly assumed that $t > s$ unless stated otherwise. As we have spatial-translation invariance in our system, we shall find that all two-point quantities depend merely on the difference $\boldsymbol{r} := \boldsymbol{x} - \boldsymbol{x}'$ of the spatial coordinates.

The response functions of the fields $\{\mathcal{O}_a(t, \boldsymbol{x})\}$ to the conjugate fields $\{h_a(t, \boldsymbol{x})\}$,

$$\mathcal{R}_{ab}(t, s, \boldsymbol{x} - \boldsymbol{x}') := \left. \frac{\delta \langle \mathcal{O}_a(t, \boldsymbol{x}) \rangle_{\{h\}}}{\delta h_b(s, \boldsymbol{x}')} \right|_{\{h\}=\{0\}}, \tag{2.19}$$

are obtained by linearly perturbing the Hamiltonion, $\mathcal{H} \to \mathcal{H} - \sum_{a,t,\boldsymbol{x}} h_a(t, \boldsymbol{x}) \mathcal{O}_a(t, \boldsymbol{x})$, as detailed in the appendix. The above-defined response function can be interpreted as the susceptibility of the expectation value of a field to near-equilibrium fluctuations.

Finally, we also obtain out-of-equilibrium responses of the fields $\{\mathcal{O}_a(t, \boldsymbol{x})\}$ to local temperature fluctuations. This we do by perturbing the noise strength $T \to T + \delta T(t, \boldsymbol{x})$ and then evaluating the response functions

$$\mathcal{R}_a^{(T)}(t, s, ; \boldsymbol{x} - \boldsymbol{x}') := \left. \frac{\delta \langle \mathcal{O}_a(t, \boldsymbol{x}) \rangle_{\delta T}}{\delta T(s, \boldsymbol{x}')} \right|_{\delta T=0}. \tag{2.20}$$

Let us specify at this point the asymptotic scaling forms that we expect for the autocorrelation function $\mathcal{C}_{ab}(t, s) := \mathcal{C}_{ab}(t, s; \mathbf{0})$ and the autoresponse functions $\mathcal{R}_{ab}(t, s) := \mathcal{R}_{ab}(t, s; \mathbf{0})$ and $\mathcal{R}_a^{(T)}(t, s) := \mathcal{R}_a^{(T)}(t, s; \mathbf{0})$. They are expected to behave as

$$\mathcal{C}_{ij}(t, s) = s^{-b_{ij}} f_C^{ij}(t/s), \qquad f_C^{ij}(y) \overset{y\to\infty}{\sim} y^{-\lambda_C^{ij}/z}, \tag{2.21}$$

$$\mathcal{R}_{ij}(t, s) = s^{-a_{ij}-1} f_R^{ij}(t/s), \qquad f_R^{ij}(y) \overset{y\to\infty}{\sim} y^{-\lambda_R^{ij}/z}, \tag{2.22}$$

$$\mathcal{R}_i^{(T)}(t, s) = s^{-a_i^{(T)}-1} f_R^{(T)i}(t/s), \qquad f_R^{(T)i}(y) \overset{y\to\infty}{\sim} y^{-\lambda_R^{(T)i}/z}, \tag{2.23}$$

in the scaling regime where t, s and $t - s$ are simultaneously large. This also defines the non-equilibrium critical exponents $a_{ij}, b_{ij}, a_i^T, \lambda_R^{ij}, \lambda_C^{ij}, \lambda_R^{(T)i}$.

We now write the correlation and response functions of some of the fields $\{\mathcal{O}_a(t, \boldsymbol{x})\}$ in terms of the spin–spin correlator $C(t, s; \boldsymbol{r})$, the constraint function $g(t; T)$ and $f(t; \boldsymbol{r})$. The details of these computations are given in the appendix, while the explicit forms of these functions and their asymptotics are spelt out in the next subsection.

2.2.1. The correlation functions.

We obtain the following expressions for the non-vanishing correlation functions of the composite fields.

- The spin^2–spin^2 correlation function is found to be

$$\mathcal{C}_{22}(t, s; \boldsymbol{r}) = \langle \mathcal{O}_2(t, \boldsymbol{r})\mathcal{O}_2(s, \boldsymbol{0})\rangle = 2[C(t, s; \boldsymbol{r})]^2. \tag{2.24}$$

For the short-range case, this formula has already been found in [28].

- The spin^2–energy-density correlation functions are

$$\mathcal{C}_{2\epsilon}(t, s; \boldsymbol{r}) = \langle \mathcal{O}_2(t, \boldsymbol{r})\mathcal{O}_\epsilon(s, \boldsymbol{0})\rangle = \frac{-1}{2g(t; T)}\partial_t(g(t; T)\mathcal{C}_{22}(t, s; \boldsymbol{r})), \tag{2.25}$$

and

$$\mathcal{C}_{\epsilon 2}(t, s; \boldsymbol{r}) = \mathcal{C}_{2\epsilon}(t, s; \boldsymbol{r}). \tag{2.26}$$

This is a stronger result than the obvious relation $\mathcal{C}_{\epsilon 2}(t, s; \boldsymbol{r}) = \mathcal{C}_{2\epsilon}(s, t; -\boldsymbol{r})$ and follows from $\omega(\boldsymbol{k}) = \omega(-\boldsymbol{k})$.

- The energy-density–energy-density correlation function is given by

$$\mathcal{C}_{\epsilon\epsilon}(t, s; \boldsymbol{r}) = \frac{-1}{2g(t; T)}\partial_t(g(t; T)\mathcal{C}_{2\epsilon}(t, s; \boldsymbol{r})). \tag{2.27}$$

2.2.2. The response functions.

For the response functions, we obtain the following expressions. Because of causality, in all expressions given below the factor $\Theta(t - s)$ is implied, where the step function $\Theta(t - s) = 1$ for $t > s$ and zero otherwise.

- Responses to the magnetic field $h_1(t, \boldsymbol{x})$, which are obtained when $\mathcal{H} \to \mathcal{H} - \sum_{t,\boldsymbol{x}} h_1(t, \boldsymbol{x})S(t, \boldsymbol{x})$, are given by

$$\mathcal{R}_{11}(t, s; \boldsymbol{r}) = \sqrt{\frac{g(s; T)}{g(t; T)}}\, f\left(\frac{t - s}{2}, \boldsymbol{r}\right), \tag{2.28}$$

$$\mathcal{R}_{21}(t, s; \boldsymbol{r}) = \mathcal{R}_{\epsilon 1}(t, s; \boldsymbol{r}) = 0. \tag{2.29}$$

- Responses to the conjugate field $h_2(t, \boldsymbol{x})$ of spin^2 operator are obtained when $\mathcal{H} \to \mathcal{H} - \sum_{t,\boldsymbol{x}} h_2(t, \boldsymbol{x})\mathcal{O}_2(t, \boldsymbol{x})$ and are given by

$$\mathcal{R}_{12}(t, s; \boldsymbol{r}) = 0, \tag{2.30}$$

$$\mathcal{R}_{22}(t, s; \boldsymbol{r}) = 4\mathcal{R}_{11}(t, s; \boldsymbol{r})C(t, s; \boldsymbol{r}), \tag{2.31}$$

$$\mathcal{R}_{\epsilon 2}(t, s; \boldsymbol{r}) = -\mathcal{R}_{22}(t, s; \boldsymbol{r})\partial_t \ln f\left(\frac{t - s}{2}, \boldsymbol{r}\right), \tag{2.32}$$

The expression for $\mathcal{R}_{22}(t, s; \boldsymbol{r})$ has already been given in [28] for the short-range model.

- Responses to the conjugate field $h_\epsilon(t, \boldsymbol{x})$ of energy-density operator are obtained when $\mathcal{H} \to \mathcal{H} - \sum_{t,\boldsymbol{x}} h_\epsilon(t, \boldsymbol{x})\mathcal{O}_\epsilon(t, \boldsymbol{x})$ and are given by

$$\mathcal{R}_{1\epsilon}(t, s; \boldsymbol{r}) = 0, \tag{2.33}$$

$$\mathcal{R}_{2\epsilon}(t, s; \boldsymbol{r}) = \frac{-1}{2g(t; T)}\partial_t(g(t; T)\mathcal{R}_{22}(t, s; \boldsymbol{r})), \tag{2.34}$$

$$\mathcal{R}_{\epsilon\epsilon}(t, s; \boldsymbol{r}) = \frac{-1}{2g(t; T)}\partial_t(g(t; T)\mathcal{R}_{2\epsilon}(t, s; \boldsymbol{r})). \tag{2.35}$$

- The spin, the spin^2 and the energy-density responses to temperature fluctuation are

$$\mathcal{R}_1^{(T)}(t, s; \boldsymbol{r}) = 0, \tag{2.36}$$

$$\mathcal{R}_2^{(T)}(t, s; \boldsymbol{r}) = 2(\mathcal{R}_{11}(t, s; \boldsymbol{r}))^2, \tag{2.37}$$

$$\mathcal{R}_\epsilon^{(T)}(t, s; \boldsymbol{r}) = \frac{-1}{2g(t; T)}\partial_t(g(t; T)\mathcal{R}_2(t, s; \boldsymbol{r})), \tag{2.38}$$

respectively.

2.3. Late-time behaviour of correlation- and response- functions

In this section, we first explicitly evaluate in the scaling limit the quantities specified in the previous subsection, and then identify the critical exponents and scaling functions. The treatment is based on previous results and techniques from [16, 20].

In the late-time limit we can approximate the function $\omega(\boldsymbol{k}) \approx B|\boldsymbol{k}|^\sigma$, where $0 < \sigma < 2$ [16]. Hence, the dynamical exponent in this range of σ is given by

$$z = \sigma. \tag{2.39}$$

Furthermore, the large-time behaviour of $f(t)$ and $g(t; T)$ are as follows. The function $f(t, \boldsymbol{x})$ in this limit becomes

$$f(t; \boldsymbol{x}) \approx B_0 t^{-d/\sigma} G(|\boldsymbol{x}|t^{-1/\sigma}); \qquad B_0 := \int_{\boldsymbol{k}} \mathrm{e}^{-2B|\boldsymbol{k}|^\sigma}. \tag{2.40}$$

Here, the scaling function $G(|\mathbf{u}|t^{-1/\sigma})$ for any variable $\mathbf{u}$ is defined as

$$G(|\mathbf{u}|t^{-1/\sigma}) := B_0^{-1} t^{d/\sigma} \int_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\mathbf{u}}\, \mathrm{e}^{-2B|\boldsymbol{k}|^\sigma t}, \tag{2.41}$$

where $\int_{\boldsymbol{k}} \cdots = (2\pi)^{-d}\int \mathrm{d}^d k \cdots$ denotes an integral over $\mathbb{R}^d$.

The Laplace transform of $f(t)$ is given by the expression

$$f_L(p) = -A_0 p^{-1+d/\sigma} + \sum_{n=1}^{\infty} A_n(-p)^{n-1}, \tag{2.42}$$

where the universal constant $A_0 = |\Gamma(1 - d/\sigma)|B_0$ and the nonuniversal constants $A_n = \int_{\Lambda_k}(2\omega_{\boldsymbol{k}})^{-n} - \int_{\boldsymbol{k}}(2B|\boldsymbol{k}|^\sigma)^{-n}$, for $n = 1, 2 \ldots$. We note that $A_1 = 1/2T_c$.

Now the constraint equation (2.11), upon Laplace transforming, becomes

$$g_L(p; T) = \frac{f_L(p)}{1 - 2T f_L(p)}, \tag{2.43}$$

and is solved in the small-p region using equation (2.42). Following a similar analysis as done for $\sigma = 2$ case in [20], we find the large-t limit of the function $g(t; T)$, which is given in equations (2.44), (2.60) and (2.76). This asymptotic constraint function has three different forms depending on the quenched temperature and the lattice dimension, for a given value of the parameter σ. The known case for the short-range model can be obtained by taking the limit $\sigma \to 2$. The three cases are

- $T < T_c$. This case was treated in [16] for the spin–spin correlator and the spin response. We recover their results and further add other correlation and response functions of the composite fields.
- $T = T_c, \sigma < d < 2\sigma$. To the best of our knowledge the quench to criticality has not been treated before. We must further distinguish two critical cases. In the first case, d can at most be 4 since $\sigma \leqslant 2$.
- $T = T_c, d > 2\sigma$. In this second case of a critical quench, the space dimension d is not bounded from above. This case includes the mean-field case.

We now discuss the large-time behaviour of the correlation and response functions in these three cases.

2.3.1. *Case I:* $T < T_c$.

Since the system exhibits space-translation invariance we take $\boldsymbol{x}' = \boldsymbol{0}$. We denote $y = t/s > 1$. The constraint function for $T < T_c$ in the large-time limit [16] is

$$g(t;T) \approx B_0 \left(1 - \frac{T}{T_c}\right)^{-1} t^{-d/\sigma}, \tag{2.44}$$

and hence the spin–spin correlation function for $T < T_c$ in the scaling regime reduces to

$$\widehat{C}(t,s;\boldsymbol{k}) = \left(1 - \frac{T}{T_c}\right) B_0^{-1} s^{d/\sigma} y^{d/2\sigma} \mathrm{e}^{-B|\boldsymbol{k}|^\sigma (t+s)}, \tag{2.45}$$

in the Fourier space or

$$C(t,s;\boldsymbol{r}) = C_0 y^{d/2\sigma} (y+1)^{-d/\sigma} G(u), \tag{2.46}$$

in the direct space, where $C_0 = 2^{d/\sigma}(1 - T/T_c)$. Here and below, expressions become shorter with the use of the three related scaling variables u, v and w, where

$$\begin{aligned} u &= |\boldsymbol{r}|((t+s)/2)^{-1/\sigma} = w(1+s/t)^{-1/\sigma}, \\ v &= |\boldsymbol{r}|((t-s)/2)^{-1/\sigma} = w(1-s/t)^{-1/\sigma}, \\ w &= |\boldsymbol{r}|(t/2)^{-1/\sigma}. \end{aligned} \tag{2.47}$$

The autocorrelation function can now be directly deduced since the scaling function $G(0) = 1$ for $\boldsymbol{r} = 0$. Hence one reads off, see (2.21) and table 1,

$$b_{11} = 0, \qquad \lambda_C^{11} = \frac{d}{2}, \qquad f_C^{11}(y) = C_0 y^{d/2\sigma} (y+1)^{-d/\sigma}. \tag{2.48}$$

Below we list the remaining expressions in the scaling limit. The autocorrelation and autoresponse functions are obtained for the composite operators in a similar way as is demonstrated for $C(t,s;\boldsymbol{r}) = \mathcal{C}_{11}(t,s;\boldsymbol{r})$. The non-equilibrium ageing exponents are listed in tables 1 and 2, for future reference.

We first list the non-vanishing correlation functions.

- The spin^2–spin^2 correlator, obtained by substituting equation (2.46) into (2.24), is

$$\mathcal{C}_{22}(t,s;\boldsymbol{r}) = 2C_0^2 y^{d/\sigma} (y+1)^{-2d/\sigma} G^2(u). \tag{2.49}$$

Table 1. Non-equilibrium exponents b, λ_C, as defined in (2.21), for several non-equilibrium autocorrelation functions in the long-range spherical model. The exponents for the short-range model can be recovered by taking the limit $\sigma \to 2$.

	b		λ_C		
	$T < T_c$	$T = T_c$	$T < T_c$	$T = T_c$	
Function				$\sigma < d < 2\sigma$	$d > 2\sigma$
$\mathcal{C}_{11}$	0	$d/\sigma - 1$	$d/2$	$3d/2 - \sigma$	d
$\mathcal{C}_{22}$	0	$2d/\sigma - 2$	d	$3d - 2\sigma$	$2d$
$\mathcal{C}_{2\epsilon}$	1	$2d/\sigma - 1$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$
$\mathcal{C}_{\epsilon\epsilon}$	2	$2d/\sigma$	$d + 2\sigma$	$3d$	$2d + 2\sigma$

Table 2. Non-equilibrium exponents $a = a'$ and λ_R, as defined in (2.22) and (2.23), for several scaling operators in the long-range spherical model. The exponents for the short-range model can be obtained by taking the limit $\sigma \to 2$.

	a		λ_R		
	$T < T_c$	$T = T_c$	$T < T_c$	$T = T_c$	
Function				$\sigma < d < 2\sigma$	$d > 2\sigma$
$\mathcal{R}_{11}$	$d/\sigma - 1$	$d/\sigma - 1$	$d/2$	$3d/2 - \sigma$	d
$\mathcal{R}_{22}$	$d/\sigma - 1$	$2d/\sigma - 2$	d	$3d - 2\sigma$	$2d$
$\mathcal{R}_{\epsilon 2}$	d/σ	$2d/\sigma - 1$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$
$\mathcal{R}_{2\epsilon}$	d/σ	$2d/\sigma - 1$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$
$\mathcal{R}_{\epsilon\epsilon}$	$d/\sigma + 1$	$2d/\sigma$	$d + 2\sigma$	$3d$	$2d + 2\sigma$
$\mathcal{R}_2^T$	$2d/\sigma - 1$	$2d/\sigma - 1$	d	$3d - 2\sigma$	$2d$
$\mathcal{R}_\epsilon^T$	$2d/\sigma$	$2d/\sigma$	$d + \sigma$	$3d - \sigma$	$2d + \sigma$

- The spin^2–energy-density correlator, obtained by using equations (2.44), (2.49) in (2.25), is

$$\mathcal{C}_{2\epsilon}(t, s; \boldsymbol{r}) = \frac{2C_0^2}{\sigma} s^{-1} y^{d/\sigma} (y+1)^{-1-2d/\sigma} G(u) D_u G(u), \tag{2.50}$$

where the operator D_z is defined as

$$D_z := z\partial_z + d. \tag{2.51}$$

- The energy-density–energy-density correlator, obtained by inserting equations (2.44), (2.50) into (2.27), is given by

$$\mathcal{C}_{\epsilon\epsilon}(t, s; \boldsymbol{r}) = \frac{C_0^2}{\sigma^2} s^{-2} y^{d/\sigma} (y+1)^{-2-2d/\sigma} (D_u + d + \sigma)[G(u) D_u G(u)]. \tag{2.52}$$

Next we write the non-vanishing response functions.

- The spin-response function, obtained using equations (2.40), (2.44) in (2.28), is given by

$$\mathcal{R}_{11}(t, s; \boldsymbol{r}) = C_1 s^{-d/\sigma} y^{d/2\sigma} (y-1)^{-d/\sigma} G(v), \tag{2.53}$$

where $C_1 = \int_{\boldsymbol{k}} \exp(-B|\boldsymbol{k}|^\sigma)$, and v was defined in equation (2.47).

- The non-vanishing response functions to spin^2 conjugate field, inferred from equations (2.31), (2.32) using (2.40), (2.46), (2.53), are given by

$$\mathcal{R}_{22}(t, s; \boldsymbol{r}) = 4C_0C_1 s^{-d/\sigma} y^{d/\sigma} (y^2-1)^{-d/\sigma} G(u) G(v), \tag{2.54}$$

$$\mathcal{R}_{\epsilon 2}(t, s; \boldsymbol{r}) = \frac{4C_0C_1}{\sigma} s^{-1-d/\sigma} y^{d/\sigma} (y^2-1)^{-d/\sigma} \frac{D_v}{y-1} G(u) G(v), \tag{2.55}$$

where D_v is as given in (2.51) and u, v were defined in (2.47).

- Responses to the energy-density conjugate field, obtained from equations (2.34), (2.35) using (2.44), (2.54), are given as follows:

$$\mathcal{R}_{2\epsilon}(t,s;\boldsymbol{r}) = \frac{2C_0C_1}{\sigma}s^{-1-d/\sigma}y^{d/\sigma}(y^2-1)^{-d/\sigma}\left(\frac{D_u}{y+1}+\frac{D_v}{y-1}\right)G(u)G(v), \tag{2.56}$$

$$\mathcal{R}_{\epsilon\epsilon}(t,s;\boldsymbol{r}) = \frac{C_0C_1}{\sigma^2}s^{-2-d/\sigma}y^{d/\sigma}(y^2-1)^{-d/\sigma} \times\left(\frac{D_u^2+\sigma D_u}{(y+1)^2}+\frac{2D_uD_v}{y^2-1}+\frac{D_v^2+\sigma D_v}{(y-1)^2}\right)G(u)G(v). \tag{2.57}$$

- The spin² and energy-density responses to local temperature fluctuations, obtained using equations (2.44), (2.53) in (2.37), (2.38), are

$$\mathcal{R}_2^{(T)}(t,s;\boldsymbol{r}) = 2C_1^2s^{-2d/\sigma}y^{d/\sigma}(y-1)^{-2d/\sigma}G^2(v), \tag{2.58}$$

$$\mathcal{R}_\epsilon^{(T)}(t,s;\boldsymbol{r}) = \frac{2C_1^2}{\sigma}s^{-1-2d/\sigma}y^{d/\sigma}(y-1)^{-1-2d/\sigma}G(v)D_vG(v), \tag{2.59}$$

respectively.

2.3.2. Case IIa: $T=T_c$ and $\sigma<d<2\sigma$.

For $T=T_c$ and $\sigma<d<2\sigma$, the constraint function has the form

$$g(t;T_c) \approx \left(4T_c^2A_0\Gamma(-1+d/\sigma)\right)^{-1}t^{-2+d/\sigma}, \tag{2.60}$$

and hence the correlation function in the scaling regime reduces to

$$\widehat{C}(t,s;\boldsymbol{k}) = 2T_csy^{1-d/2\sigma}\int_0^1 \mathrm{d}z\,\mathrm{e}^{-B|\boldsymbol{k}|^\sigma(t+s-2sz)}z^{-2+d/\sigma}, \tag{2.61}$$

while in direct space it is given by

$$C(t,s;\boldsymbol{r}) = 2T_cC_1s^{1-d/\sigma}y^{1-d/2\sigma}(y+1)^{-d/\sigma}\sum_{n=0}^{\infty}\frac{(y+1)^{-n}G_n(u)}{n!(n-1+d/\sigma)}, \tag{2.62}$$

where u is given in (2.47) and the function $G_n(|\mathfrak{v}|t^{-1/\sigma})$ is defined as

$$G_n(|\mathfrak{v}|t^{-1/\sigma}) := 4^nt^{n+d/\sigma}B_0^{-1}\int_{\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\mathfrak{v}}\,\mathrm{e}^{-2B|\boldsymbol{k}|^\sigma t}(B|\boldsymbol{k}|^\sigma)^n, \tag{2.63}$$

for any variable $\mathfrak{v}$. The spin-response function in this case has the form

$$\mathcal{R}_{11}(t,s;\boldsymbol{x}) = C_1s^{-d/\sigma}y^{1-d/2\sigma}(y-1)^{-d/\sigma}G(v). \tag{2.64}$$

To avoid presenting lengthy expressions we write only the leading behaviour in y for the correlators and responses in this case. The spin–spin correlation function in this approximation becomes

$$C(t,s;\boldsymbol{r}) \approx 2\widetilde{T}_cC_1s^{1-d/\sigma}y^{1-3d/2\sigma}G(w), \tag{2.65}$$

where $\widetilde{T}_c = T_c\sigma/(d-\sigma)$, and w is as given in (2.47). Setting w and v to zero, we can read off the ageing exponents, see tables 1 and 2,

$$a_{11} = b_{11} = \frac{d}{\sigma}-1, \qquad \lambda_R^{11} = \lambda_C^{11} = \frac{3d}{2}-\sigma, \qquad z=\sigma \tag{2.66}$$

The other non-vanishing correlators and responses are given as follows, wherein we first list the correlation functions:

- The spin^2–spin^2 correlator, obtained from equations (2.65), (2.24), is given by

$$\mathcal{C}_{22}(t,s;\boldsymbol{r}) \approx 8\widetilde{T}_c^2 C_1^2 s^{2-2d/\sigma} y^{2-3d/\sigma} G^2(w). \tag{2.67}$$

- For the spin^2–energy correlator, using (2.60), (2.67) in (2.25), we obtain

$$\mathcal{C}_{2\epsilon}(t,s;\boldsymbol{r}) \approx \frac{8\widetilde{T}_c^2 C_1^2}{\sigma} s^{1-2d/\sigma} y^{1-3d/\sigma} G(w) D_w G(w). \tag{2.68}$$

- Finally the energy–energy correlator, using (2.60), (2.68) in (2.27), reads

$$\mathcal{C}_{\epsilon\epsilon}(t,s;\boldsymbol{r}) \approx \frac{4\widetilde{T}_c^2 C_1^2}{\sigma^2} s^{-2d/\sigma} y^{-3d/\sigma} (D_w + d + \sigma)[G(w) D_w G(w)]. \tag{2.69}$$

The non-vanishing response functions are listed below.

- The responses to the spin^2 conjugate field, obtained using (2.40), (2.64), (2.65) in (2.31), (2.32), are given by

$$\mathcal{R}_{22}(t,s;\boldsymbol{r}) \approx 8\widetilde{T}_c C_1^2 s^{1-2d/\sigma} y^{2-3d/\sigma} G^2(w), \tag{2.70}$$

$$\mathcal{R}_{\epsilon 2}(t,s;\boldsymbol{r}) \approx \frac{8\widetilde{T}_c C_1^2}{\sigma} s^{-2d/\sigma} y^{1-3d/\sigma} G(w) D_w G(w). \tag{2.71}$$

- The responses to energy-density conjugate field, obtained from (2.60), (2.70) and (2.34), (2.35), are

$$\mathcal{R}_{2\epsilon}(t,s;\boldsymbol{r}) \approx \mathcal{R}_{\epsilon 2}(t,s;\boldsymbol{r}), \tag{2.72}$$

$$\mathcal{R}_{\epsilon\epsilon}(t,s;\boldsymbol{r}) \approx \frac{4\widetilde{T}_c C_1^2}{\sigma^2} s^{-1-2d/\sigma} y^{-3d/\sigma} (D_w + d + \sigma)[G(w) D_w G(w)]. \tag{2.73}$$

- Lastly, the responses to temperature fluctuations, obtained from (2.60), (2.64) and (2.37), (2.38), are

$$\mathcal{R}_2^{(T)}(t,s;\boldsymbol{r}) \approx 2C_1^2 s^{-2d/\sigma} y^{2-3d/\sigma} G^2(w), \tag{2.74}$$

$$\mathcal{R}_\epsilon^{(T)}(t,s;\boldsymbol{r}) \approx \frac{2C_1^2}{\sigma} s^{-1-2d/\sigma} y^{1-3d/\sigma} G(w) D_w G(w). \tag{2.75}$$

2.3.3. *Case IIb: $T = T_c$ and $d > 2\sigma$.*

For $T = T_c$ and $d > 2\sigma$, the constraint function at large times is

$$g(t;T_c) \approx \left(4T_c^2 A_2\right)^{-1}. \tag{2.76}$$

This is just a constant and does not appear in the correlation and response functions to leading order in this large-time limit. In this case, the correlation function in the scaling regime reduces to

$$\widehat{C}(t,s;\boldsymbol{k}) = \frac{T_c}{B|\boldsymbol{k}|^\sigma} \left(\mathrm{e}^{-B|\boldsymbol{k}|^\sigma (t-s)} - \mathrm{e}^{-B|\boldsymbol{k}|^\sigma (t+s)}\right), \tag{2.77}$$

and in the direct space it is

$$C(t,s;\boldsymbol{r}) = 2T_c C_1 s^{1-d/\sigma} \left(\frac{G_{-1}(v)}{(y-1)^{d/\sigma-1}} - \frac{G_{-1}(u)}{(y+1)^{d/\sigma-1}} \right), \tag{2.78}$$

where G_{-1} is as given in (2.63).

The spin-response function in this case is given by

$$\mathcal{R}_{11}(t,s;\boldsymbol{r}) = C_1 s^{-d/\sigma}(y-1)^{-d/\sigma}G(v). \tag{2.79}$$

Here, again we present only the leading behaviour in y of the correlators and responses. The correlation function in this approximation becomes

$$C(t,s;\boldsymbol{r}) \approx 2T_c s f(t/2,\boldsymbol{r}) = 2T_c C_1 s^{1-d/\sigma} y^{-d/\sigma} G(w). \tag{2.80}$$

Again we read off the critical exponents after setting $v = w = 0$

$$a_{11} = b_{11} = \frac{d}{\sigma} - 1, \qquad \lambda_R^{11} = \lambda_C^{11} = d. \tag{2.81}$$

The other non-vanishing correlation functions are given as follows:

- The spin^2–spin^2 correlation function, substituting (2.80) in (2.24), is

$$\mathcal{C}_{22}(t,s;\boldsymbol{r}) \approx 8T_c^2 C_1^2 s^{2-2d/\sigma} y^{-2d/\sigma} G^2(w). \tag{2.82}$$

- The spin^2–energy correlation function, from (2.76), (2.82), (2.25),

$$\mathcal{C}_{2\epsilon}(t,s;\boldsymbol{r}) \approx \frac{8T_c^2 C_1^2}{\sigma} s^{1-2d/\sigma} y^{-1-2d/\sigma} G(w) D_w G(w). \tag{2.83}$$

- The energy-density–energy-density correlation function, from (2.76), (2.83), (2.27), is

$$\mathcal{C}_{\epsilon\epsilon}(t,s;\boldsymbol{r}) \approx \frac{4T_c^2 C_1^2}{\sigma^2} s^{-2d/\sigma} y^{-2-2d/\sigma}(D_w + d + \sigma)[G(w) D_w G(w)]. \tag{2.84}$$

The remaining non-vanishing response functions follow.

- The responses to spin^2 conjugate field, obtained from (2.40), (2.79), (2.80) and (2.31), (2.32), are

$$\mathcal{R}_{22}(t,s;\boldsymbol{r}) \approx 8T_c C_1^2 s^{1-2d/\sigma} y^{-2d/\sigma} G^2(w), \tag{2.85}$$

$$\mathcal{R}_{\epsilon 2}(t,s;\boldsymbol{r}) \approx \frac{8T_c C_1^2}{\sigma} s^{-2d/\sigma} y^{-1-2d/\sigma} G(w) D_w G(w). \tag{2.86}$$

- The responses to energy-density conjugate field, obtained from (2.76), (2.85) and (2.34), (2.35), are given by

$$\mathcal{R}_{2\epsilon}(t,s;\boldsymbol{r}) \approx \mathcal{R}_{\epsilon 2}(t,s;\boldsymbol{r}), \tag{2.87}$$

$$\mathcal{R}_{\epsilon\epsilon}(t,s;\boldsymbol{r}) \approx \frac{4T_c C_1^2}{\sigma^2} s^{-1-2d/\sigma} y^{-2-2d/\sigma}(D_w + d + \sigma)[G(w) D_w G(w)]. \tag{2.88}$$

- Finally, the responses to temperature fluctuations, obtained from (2.76), (2.79) and (2.37), (2.38), are given as

$$\mathcal{R}_2(T)(t,s;\boldsymbol{r}) \approx 2C_1^2 s^{-2d/\sigma} y^{-2d/\sigma} G^2(w), \tag{2.89}$$

$$\mathcal{R}_\epsilon^{(T)}(t,s;\boldsymbol{r}) \approx \frac{2C_1^2}{\sigma} s^{-1-2d/\sigma} y^{-1-2d/\sigma} G(w) D_w G(w). \tag{2.90}$$

The exponents of these functions, derived in this section, are collected in tables 1 and 2.

2.3.4. Fluctuation–dissipation ratios.

An important quantity, in particular for the case of critical dynamics, is the fluctuation–dissipation ratio of an observable, which is defined as [26, 27]

$$X_{ab}(t,s) := T_c \mathcal{R}_{ab}(t,s;\mathbf{0}) \left(\frac{\partial C_{ab}(t,s;\mathbf{0})}{\partial s} \right)^{-1} \tag{2.91}$$

and its limit value

$$X_{ab}^{\infty} := \lim_{s\to\infty} (\lim_{t\to\infty} X_{ab}(t,s)) = \lim_{y\to\infty} (\lim_{s\to\infty} X_{ab}(t,s)|_{y=t/s}). \tag{2.92}$$

For case I, that is for phase-ordering kinetics, it was already known that in the quasi-static limit $s \to \infty$ but $t-s$ fixed and $\ll s$, the fluctuation–dissipation theorem still holds [16]. On the other hand, we obtain in the scaling limit $s \to \infty$ and $y = t/s > 1$ fixed that, for all observables considered here,

$$X_{11}(t,s) = X_{22}(t,s) = X_{2\epsilon}(t,s) = X_{\epsilon 2}(t,s) = X_{\epsilon\epsilon}(t,s) = \frac{2\sigma T C_1}{d C_0} s^{1-d/\sigma}. \tag{2.93}$$

For $d > \sigma$ we have therefore in this case that

$$X_{11}^{\infty} = X_{22}^{\infty} = X_{2\epsilon}^{\infty} = X_{\epsilon 2}^{\infty} = X_{\epsilon\epsilon}^{\infty} = 0 \tag{2.94}$$

as expected for a low-temperature phase (recall that for $d \leqslant \sigma$ the critical temperature is zero [24]).

In the case of critical dynamics (cases IIa and IIb), the limit fluctuation–dissipation ratios are universal numbers characterizing the critical system [20]. For their calculation, we can use directly the scaling limit $s \to \infty$ with $y = t/s$ being kept fixed. In case IIa, it is convenient to obtain the autocorrelation function $C(t,s)$ by directly integrating equation (2.61), which leads to

$$C(t,s) = \frac{2T_c C_1 \sigma}{d-\sigma} s^{1-d/\sigma} y^{1-d/2\sigma} (y-1)^{1-d/\sigma} (y+1)^{-1}. \tag{2.95}$$

Combining this with equation (2.64), we get

$$X_{11}(t,s) = X_{11}(y) = \frac{1}{2}(y+1) \left[1 + \frac{y-1}{d-\sigma} \left(\frac{d}{2} - \frac{\sigma}{y+1} \right) \right]^{-1}. \tag{2.96}$$

Similarly, in case IIb, using equations (2.78) and (2.79), and upon substituting the value of $G_{-1}(0) = \sigma G(0)/(d-\sigma)$, we find

$$X_{11}(t,s) = X_{11}(y) = \left(1 + \left(\frac{y-1}{y+1} \right)^{d/\sigma} \right)^{-1} \tag{2.97}$$

In particular, we see that in the quasi-static limit $s \to \infty$ with $t-s$ being kept fixed (or alternatively $y \to 1$), $\lim_{y\to 1} X_{11}(y) \to 1$ in both critical cases, such that the fluctuation–dissipation theorem holds. Similarly, from the relations (2.24), (2.25), (2.27) and (2.31), (2.34), (2.35) we also have $\lim_{y\to 1} X_{22}(y) = \lim_{y\to 1} X_{\epsilon\epsilon}(y) = \lim_{y\to 1} X_{2\epsilon}(y) = 1$. On the other hand, and remarkably, the limit fluctuation–dissipation ratio turns out to be independent of the choice of the considered observable. We find for $y \to \infty$

$$X_{11}^{\infty} = X_{22}^{\infty} = X_{2\epsilon}^{\infty} = X_{\epsilon 2}^{\infty} = X_{\epsilon\epsilon}^{\infty} = \begin{cases} 1 - \sigma/d & \text{for the case IIa} \\ 1/2 & \text{for the case IIb.} \end{cases} \tag{2.98}$$

This reduces to the well-known expressions in the short-range model [20] when $z = \sigma \to 2$. We recall that in [28] a slightly different definition for the energy density was used, in which case the value for the corresponding fluctuation–dissipation ratio may be different.

3. Local scale invariance

The theory of local scale invariance (LSI) was developed in a series of papers [6, 7, 14, 29, 30], using local symmetries to fix the response and correlation functions. For recent reviews which focus on different types of applications see [11–13]. For our purposes here it is sufficient to just quote a few results. A central concept of LSI are the *quasi-primary* scaling operators [7], which transform in the simplest possible way under local-scale transformations, very much in analogy with the (quasi)primary scaling operators of conformal field theory [10].[7] A quasi-primary scaling operator ϕ is characterized by a set of 'quantum numbers' (x, ξ, μ, β), where x is the 'scaling dimension' of ϕ and μ is sometimes referred to as the 'mass' of ϕ (not to be confused with the lattice constant μ in section 2.1).

3.1. Response functions

For a given dynamical exponent z, LSI yields the following prediction for the response function of a quasi-primary operator ϕ characterized by the parameters (x, ξ, μ, β) [7, 13, 14, 17]:

$$\begin{aligned} R^{\mathrm{LSI}}(t, s; \boldsymbol{r}) &= \delta_{\mu,-\tilde{\mu}}\delta_{\beta,\tilde{\beta}} R(t, s)\mathcal{F}^{(\mu,\beta)}\left(\frac{|\boldsymbol{r}|}{(t-s)^{1/z}}\right), \\ R(t, s) &= s^{-1-a}\left(\frac{t}{s}\right)^{1+a'-\lambda_R/z}\left(\frac{t}{s}-1\right)^{-1-a'}, \end{aligned} \tag{3.1}$$

where the exponents a, a' and λ_R are related to the parameters (x, ξ, μ) via

$$a+1 = \frac{1}{z}(x+\tilde{x}), \qquad a'+1 = \frac{1}{z}(x+2\xi+\tilde{x}+2\tilde{\xi}), \qquad \frac{\lambda_R}{z} = \frac{2x}{z}+\frac{2\xi}{z}, \tag{3.2}$$

and the parameters $(\tilde{x}, \tilde{\xi}, \tilde{\mu}, \tilde{\beta})$ characterize the response field $\tilde{\phi}$. The spacetime part $\mathcal{F}^{(\mu,\beta)}(\rho)$ (where $\rho := |\boldsymbol{\rho}|$ and $\boldsymbol{\rho} = \boldsymbol{r}(t-s)^{-1/\sigma}$) satisfies the following fractional differential equation:

$$\left(\partial_\rho + z\mu\rho\partial_\rho^{2-z} + [\beta\mu + \mu(2-z)]\partial_\rho^{1-z}\right)\mathcal{F}^{(\mu,\beta)}(\rho) = 0, \tag{3.3}$$

which also illustrates that the 'mass' μ may be interpreted as a generalized diffusion constant. The fractional derivatives ∂_ρ^α are defined and discussed in [7]. Recall, however, that the definition used here is not unique and that different non-equivalent definitions for fractional derivatives exist [31, 32]. If $z = N + p/q$, where $N = [z]$ is the largest integer less or equal to z, $0 \leqslant p/q < 1$ and p and q coprime, the solution of (3.3) by series methods is particularly simple, with the result [14]

$$\mathcal{F}^{(\mu,\beta)}(\rho) = \sum_{m\in\mathcal{E}} c_m \phi^{(m)}(\rho), \qquad \text{with} \qquad \phi^{(m)}(\rho) = \sum_{n=0}^{\infty} b_n^{(m)} \rho^{(n-1)z+p/q+m+1}. \tag{3.4}$$

The constants c_m are not determined by LSI and the set $\mathcal{E}$ is

$$\mathcal{E} = \begin{cases} -1, 0, \ldots, N-1 & p \neq 0 \\ 0, \ldots, N-1, & p = 0. \end{cases} \tag{3.5}$$

Finally, the coefficients $b_n^{(m)}$ read

$$b_n^{(m)} = \frac{(-z^2\mu)^n\Gamma(p/q+1+m)\Gamma(n+z^{-1}(p/q+m)+\beta+2-z)}{\Gamma((n-1)z+p/q+m+2)\Gamma(z^{-1}(p/q+m)+\beta+2-z)}, \tag{3.6}$$

such that $\phi^{(m)}(\rho)$ has an infinite radius of convergence for $z > 1$.

[7] Specifically, if $\mathcal{X}$ is an infinitesimal generator of a local-scale transformation and ϕ a quasi-primary scaling operator, $\delta\phi = -\varepsilon\mathcal{X}\phi$. Usually, the order parameter corresponds to a quasi-primary operator, but if ϕ is quasi-primary, then neither $\partial_t\phi$ nor $\partial_{\vec{r}}\phi$ are. The n-point functions $\langle\phi_1 \ldots \phi_n\rangle$ of quasi-primary operators transform covariantly and hence satisfy linear differential equations $\mathcal{X}^{[n]}\langle\phi_1 \ldots \phi_n\rangle = 0$.

Let us now consider the magnetic response of the order parameter, $\mathcal{R}_{11}$, the result for which we recall from (2.28) is

$$\mathcal{R}_{11}(\boldsymbol{r};t,s) = (2\pi)^d s^{-d/\sigma}\left(\frac{t}{s}\right)^{-F/2}(t/s-1)^{-d/\sigma}\int_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}(t-s)^{-1/\sigma}}\,\mathrm{e}^{-Bk^{\sigma}}$$
$$= R(t,s)\sum_{n=0}^{\infty} a_n\rho^{2n}, \qquad \boldsymbol{\rho} = \boldsymbol{r}(t-s)^{-1/\sigma}, \tag{3.7}$$

where the exponent F is given by

$$F = \begin{cases} -d/\sigma & \text{case I} \\ -2+d/\sigma & \text{case IIa} \\ 0 & \text{case IIb} \end{cases}. \tag{3.8}$$

Clearly, the spacetime part of the LSI prediction does *not* agree with this result since the exponents of ρ in equations (3.7) and (3.4) are linearly independent if z is not an integer. In equation (3.7), we have expanded the exponential in order to rewrite this as a series in $\rho = |\boldsymbol{\rho}|$. This form of the series is incompatible with the expected form (3.4) for $z < 2$. This disagreement has motivated us to look for a new formulation of LSI, which uses a more appropriate form of fractional derivatives ∇_r^{α}. This formulation including the exact definition of ∇_r^{α} will be described elsewhere in detail [17], here we only mention two results we need:

1. *Generalized Bargmann superselection rule*. Let a system be given with dynamical exponent $z \neq \frac{2k+2}{2k+1}$, $(k \in \mathbb{N})$. Let $\{\phi_i\}$ be a set of quasi-primary scaling operators, each characterized by the set $(x_i, \xi_i, \mu_i, \beta_i)$. Then the $(2n)$-point function

$$F^{(2n)} := \langle \phi_1(t_1,\boldsymbol{r}_1)\ldots\phi_{2n}(t_{2n},\boldsymbol{r}_{2n})\rangle \tag{3.9}$$

is zero unless μ_i form n distinct pairs $(\mu_i, \mu_{\tau(i)})$ $(i = 1,\ldots n)$, such that

$$\mu_i = -\mu_{\tau(i)}. \tag{3.10}$$

2. The decomposition (3.1) of the response function remains valid, but its spacetime part now satisfies the fractional differential equation, which is quite similar to equation (3.3)

$$\left(\partial_\rho + z\mu\rho\nabla_\rho^{2-z} + [\beta\mu + \mu(2-z)]\partial_\rho\nabla_\rho^{-z}\right)\mathcal{F}^{(\mu,\beta)}(\rho) = 0. \tag{3.11}$$

A solution of equation (3.11) reads [17]

$$\mathcal{F}^{(\mu,\beta)}(\boldsymbol{\rho}) = f_0\int_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{\rho}\cdot\boldsymbol{k}}|\boldsymbol{k}|^{\beta}\exp\left(-\frac{1}{z^2\mathrm{i}^{2-z}\mu}|\boldsymbol{k}|^z\right) \tag{3.12}$$

We see that this prediction of the 'new' formulation of LSI is fully compatible with our exact result (3.7) for $\mathcal{R}_{11}(t,s;\boldsymbol{r})$ if we identify

$$\mu_1 = -\tilde{\mu}_1 = (z^2B\mathrm{i}^{2-z})^{-1}, \qquad \beta_1 = \tilde{\beta}_1 = 0, \qquad g_0 = (2\pi)^d, \tag{3.13}$$

and set for the critical exponents

$$a_{11} = a'_{11} = \frac{d}{\sigma} - 1, \qquad \lambda_R^{11} = d + \frac{\alpha\sigma}{2}. \tag{3.14}$$

This agreement supports the assumption that the fields ϕ and $\widetilde{\phi}$ are both quasi-primary with $\mu = -\tilde{\mu}$ and $\beta = \tilde{\beta}$. This is further supported by the fact that $\mathcal{R}_{12}(t,t';\boldsymbol{r}) = 0 = \mathcal{R}_{1\epsilon}(t,t';\boldsymbol{r})$, which is predicted by LSI because of the generalized Bargmann superselection rule.

Having verified that the response function for the order-parameter field ϕ agrees with LSI, and thus having confirmed that ϕ is indeed quasi-primary, we now enquire whether this holds for composite operators. First, we consider the short-range model $\sigma \geqslant 2$. The relevant results can be read from those of section 2 if we let $\sigma \to 2$. Then the response $\mathcal{R}_{11}(t, s; \boldsymbol{r})$ in equation (3.7) simplifies to

$$\mathcal{R}_{11}(t, s; \boldsymbol{r}) = s^{-d/2} \left(\frac{t}{s}\right)^{-F/2} \left(\frac{t}{s} - 1\right)^{-d/2} \exp\left(-\frac{1}{4B} \frac{\boldsymbol{r}^2}{t-s}\right), \tag{3.15}$$

up to a normalization constant. Similarly, the temperature response of the spin^2 field, from the above expression and equation (2.37), becomes

$$\mathcal{R}_2^{(T)}(t, s; \boldsymbol{r}) = s^{-d} \left(\frac{t}{s}\right)^{-F} \left(\frac{t}{s} - 1\right)^{-d} \exp\left(-\frac{1}{2B} \frac{\boldsymbol{r}^2}{t-s}\right), \tag{3.16}$$

which is of the form predicted by equation (3.12), if we identify

$$\mu_2 = -\tilde{\mu}_2 = 2\mu_1, \qquad \beta_2 = \tilde{\beta}_2 = 0, \tag{3.17}$$

and

$$a_{22} = a'_{22} = 2a_{11} + 1, \qquad \lambda_R^{22} = 2\lambda_R^{11}. \tag{3.18}$$

Physically, we can therefore identify temperature changes as the conjugate variable to the spin-squared operator, at least for the short-ranged case. On the other hand, the spin^2 response $\mathcal{R}_{22}$ to the perturbation $h_2(t, \boldsymbol{x})$ cannot be cast into that form. This can easily be seen in equation (2.54), which has a dependence on $t + s$, while the LSI-predicted form does not contain this dependence. Note that this response function in a field-theoretical setting (see, for example, [8, 9]) corresponds to $\langle \phi^2(t, \boldsymbol{x})(\phi\tilde{\phi})(s, \boldsymbol{x} + \boldsymbol{r})\rangle$.

Our findings suggest that for the short-range model the operator ϕ^2, corresponding to spin^2, is quasi-primary and so is the corresponding response field $\tilde{\phi}^2$ (obtained by locally perturbing the temperature). The parameters of these two fields are related to the fields ϕ and $\tilde{\phi}$ in the following way: if ϕ has the parameters (x, ξ, μ, β) then the parameters of ϕ^2 can be obtained from these by multiplying each parameter by the factor 2. Similarly, the parameters of $\tilde{\phi}^2$ are related to those of $\tilde{\phi}$. On the other hand, we see that the composite operator $\phi\tilde{\phi}$ (defined by a perturbation of the external field $h_2(t, \boldsymbol{x})$) is *not* quasi-primary, and neither is the energy-density operator $\epsilon(\boldsymbol{x})$, even in the short-range model (that last finding is not surprising, since we have already seen in section 2 that $\epsilon(\boldsymbol{x})$ is related to the gradient of ϕ)[8].

We now proceed to the long-range model, where $0 < \sigma < 2$. $\mathcal{R}_{22}$ cannot be brought into the LSI-predicted form, for the same reason as mentioned above for the short-range model, namely by comparing the $t + s$ dependence. The response function $\mathcal{R}_2^{(T)}$ cannot be brought into the LSI-predicted form either, since it contains a product of the type $\mathcal{F}^{(\mu,\beta)}(t, s; \boldsymbol{r})^2$. This again cannot be cast into the general form (3.12), except for $z = 2$. In this exceptional case, the special properties of a Gaussian integral ensure that $\mathcal{F}^{(\mu,\beta)}(t, s; \boldsymbol{r})^2$ can be rewritten in the form (3.12) upon redefinition of parameters. By a similar analysis we find that $\mathcal{R}_{\epsilon 2}$ does not have the LSI-predicted form. We conclude that the operator ϕ^2 is not quasi-primary under LSI for the long-range model, unlike for the short-range case $\sigma \geqslant 2$.

In a similar way, we also find that the response functions of the operator $\mathcal{O}_\epsilon$, namely $\mathcal{R}_{\epsilon 2}$ and $\mathcal{R}_{\epsilon\epsilon}$, also do not have the forms (3.1) and (3.12).

Summarizing, we have seen that in the long-range model the above composite fields, though made of quasi-primary fields, are not quasi-primary. For the time being, the order-parameter ϕ and the associate response field $\widetilde{\phi}$ related to a magnetic perturbation remain the

[8] In the Landau–Ginzbourg classification of primary scaling operators in the minimal models of $2D$ conformal field-theory (Ising, Potts, etc), one usually has that ϕ and eventually a finite number of normal-ordered powers $: \phi :^\ell$ are primary.

only scaling operators with a simple transformation under local-scale transformations. This is distinct from the short-range case of $z = 2$. It remains an open question in which sense the transformation of, say, ϕ^2 is distinct from the one of ϕ. On the other hand, the generalized Bargman superselection rule (which follows from the weaker Galilei-invariance alone) has been confirmed in all cases, by assigning the following (relative) 'masses' to the fields

$$\mu_\phi = \mu, \qquad \mu_{\mathcal{O}_2} = 2\mu, \qquad \mu_{\mathcal{O}_\epsilon} = 2\mu, \tag{3.19}$$

and with negative masses to the corresponding response fields. This is natural because of the linear structure of the theory.

3.2. Correlation functions

In this section, we compare the LSI prediction for the correlation function of the quasi-primary operator $\phi(t, \boldsymbol{x})$ with our exact result, see (2.46), (2.61), (2.77).

The LSI prediction for the correlation function, for fully disordered initial conditions with white noise, is [17]

$$C^{\rm LSI}(t, s; \boldsymbol{r}) = C^{\rm LSI}_{\rm init}(t, s; \boldsymbol{r}) + C^{\rm LSI}_{\rm th}(t, s; \boldsymbol{r}), \tag{3.20}$$

with the 'initial' part

$$C^{\rm LSI}_{\rm init}(t, s; \boldsymbol{r}) = c_0 s^{-b_{\rm init}+2\beta/z+d/z} y^{-b_{\rm init}+\lambda_R/z+\beta/z} (y-1)^{b_{\rm init}+d/z-2\lambda_R/z} \times \int_{\boldsymbol{k}} |\boldsymbol{k}|^{2\beta} \exp\left(-\frac{|\boldsymbol{k}|^z}{z^2\mu {\rm i}^{2-z}}(t+s)\right) {\rm e}^{{\rm i}\boldsymbol{r}\cdot\boldsymbol{k}}, \tag{3.21}$$

and the 'thermal' part

$$\begin{aligned} C^{\rm LSI}_{\rm th}(t, s; \boldsymbol{r}) = {} & 2T s^{-b_{\rm th}+2\beta/z+d/z} y^{2\xi/z} (y-1)^{2(1+a')-2\lambda_R/z-4\xi/z} \\ & \times \int_0^1 {\rm d}\theta\, (y-\theta)^{-2(a'+1)+\lambda_R/z+2\xi/z+\beta/z+d/z} (1-\theta)^{-2(a'+1)+\lambda_R/z+2\tilde{\xi}/z+\beta/z+d/z} \\ & \times \theta^{4\tilde{\xi}/z} g\left(\frac{1}{y}\frac{y-\theta}{1-\theta}\right) \int_{\boldsymbol{k}} |\boldsymbol{k}|^{2\beta} \exp\left(-\frac{|\boldsymbol{k}|^z s(y+1-2\theta)}{z^2\mu {\rm i}^{(2-z)}}\right) {\rm e}^{{\rm i}\boldsymbol{r}\cdot\boldsymbol{k}}. \end{aligned} \tag{3.22}$$

Here, the function $g(u)$ is not determined by the dynamical symmetries and ξ and $\tilde{\xi}$ can be considered as free parameters.

In case I, the spin–spin correlation function (2.46) can be rewritten as

$$C(t, s; \boldsymbol{r}) = s^{-F} y^{-F/2} \int_{\boldsymbol{k}} {\rm e}^{{\rm i}\boldsymbol{r}\cdot\boldsymbol{k}}\, {\rm e}^{-B|\boldsymbol{k}|^\sigma (t+s)}, \tag{3.23}$$

up to a normalization constant, with α given by (3.8). In this case ($T < T_c$), the contribution coming from the initial noise is the relevant one [1], and therefore we should compare with the spin–spin correlator $C^{\rm LSI}_{\rm init}(t, s; \boldsymbol{r})$. Indeed we find for the choice of parameters as given in (3.13), (3.14) and $b_{\rm init} = 0$ that $C^{\rm LSI}_{\rm init}(t, s; \boldsymbol{r}) = C(t, s, \boldsymbol{r})$, as it should be.

In cases IIa and IIb, the correlation function, as given in (2.61) and (2.77), can be rewritten in direct space as follows, using again (3.8) and up to normalization constant,

$$C_{\rm th}(t, s; \boldsymbol{r}) = 2Tsy^{-F/2} \int_0^1 {\rm d}\theta\, \theta^F \int_{\boldsymbol{k}} {\rm e}^{-B|\boldsymbol{k}|^\sigma s(y+1-2\theta)}\, {\rm e}^{{\rm i}\boldsymbol{r}\cdot\boldsymbol{k}}. \tag{3.24}$$

For the cases IIa ($T = T_c, \sigma < d < 2\sigma$) and IIb ($T = T_c, d > 2\sigma$), in the LSI-prediction the term coming from the thermal noise is the relevant one [1, 3]. If we set $g(u) = 1$ and, in addition to the given choice of parameters (3.13) and (3.14), let

$$b_{\rm th} = \frac{d}{z} - 1 \qquad \text{and} \qquad \xi = -\frac{1}{4}zF, \qquad \tilde{\xi} = \frac{1}{4}zF, \tag{3.25}$$

we find agreement of the LSI-predicted correlation function $C^{\rm LSI}_{\rm th}(t, s; \boldsymbol{r}) = C(t, s; \boldsymbol{r})$.

4. Conclusion

We have analysed the kinetics of the spherical model with long-range interactions when quenched onto or below the critical point T_c. For $T < T_c$ we have reproduced the results of Cannas *et al* [16] for the order parameter and for $T = T_c$ we have derived exact results for the response and correlation function of the order parameter. We also considered, for $T \leqslant T_c$, various composite fields and derived their ageing exponents and scaling functions as listed in section 2. We then have carried out a detailed test of local scale invariance using our analytical results. For this purpose, the long-range spherical model offers the useful feature that its dynamical exponent $z = \sigma$ depends continuously on one of the control parameters.

We have obtained the following results:

(i) Dynamical scaling holds for various composite fields for quenches onto or below the critical temperature. The non-equilibrium exponents are given in tables 1 and 2. The scaling functions also have been determined.

(ii) In the kinetic spherical model with short-ranged interactions ($\sigma > 2$ and hence $z = 2$), apart from the order-parameter field ϕ, its square too appears to be a quasi-primary scaling operator, as tested through several two-time response and correlation functions.

(iii) In the long-range spherical model, the first tests of the spacetime response in a system with a tuneable dynamical exponent have been performed. This shows that the formulation of LSI with $z \neq 2$, which we proposed earlier [7], even with the recent improvements given in [14], does not describe the exact result for $\mathcal{R}_{11}$ when $0 < z < 2$, although that formulation did pass previous tests when $z = 2$ [33] or $z = 4$ [14, 15].

(iv) As can be seen from the fractional differential equation satisfied by the spacetime response function, the precise definition of the fractional derivative used is crucial. We shall present elsewhere a systematic construction of new generators of local scale invariance [17] where we shall also show that *all* previous tests where $z = 2$ or $z = 4$ are passed by the new formulation. Here, we have seen that the exact results from the long-range spherical model are completely consistent with the new formulation of local scale invariance.

(v) In contrast to the short-range case where $z = 2$, the spin-squared field in the long-range model is no longer described by a quasi-primary scaling operator. This calls for a more systematic analysis, since it indicates that there might be new ways, not readily realized in conformal invariance, of non-quasi-primary scaling operators.

(vi) Both the two-time response and the correlation function of the order-parameter field ϕ are fully compatible with local scale invariance in the entire range $0 < z = \sigma < 2$.

While the analytical results presented here certainly provide useful information, the eventual confirmation of local scale invariance might appear fairly natural since the underlying Langevin equation is *linear*. Indeed, for linear Langevin equations there is a direct proof of local scale invariance which uses a decomposition of the Langevin equation into a 'deterministic part' for which non-trivial local scale-symmetries can be mathematically proven and a 'noise part' [13, 17, 29]. For nonlinear Langevin equations the formal proof of non-trivial symmetries of the 'deterministic part' is still difficult, although progress has been made [34]. In the absence of exact solutions for models described in terms of nonlinear Langevin equations numerical tests going beyond merely checking the autoresponse function $R(t, s; \mathbf{0})$ will be required and it will be useful to be able to vary the value of the dynamical exponent z. In this context, a natural candidate for such studies is the disordered Ising model quenched to $T < T_c$, where it is already known that z depends continuously on the disorder and on temperature, see [12, 35, 36] and references therein. Furthermore, its Langevin equation is nonlinear. We hope to be able soon to report tests on the spacetime behaviour of response and correlators in this model which should provide useful information on whether LSI with $z \neq 2$ can really be extended beyond the simple solvable systems studied so far.

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Appendix. Correlations and responses

We briefly present the calculational details that lead to the expressions given in section 2.2. In evaluating the expectation value of the composite operators we use Wick's contraction, which is applicable in our model if, apart from the noise, the initial spin distribution for $S(0, \boldsymbol{x})$ is also Gaussian [37]. By Wick's contraction and Fourier transforming, we get

$$\begin{aligned}\mathcal{C}_{2\epsilon}(t, t'; \boldsymbol{x}-\boldsymbol{x}') &= \int_{\Lambda_k,\Lambda_{k'}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot(\boldsymbol{x}-\boldsymbol{x}')}(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})C(t, t'; \boldsymbol{k})C(t, t'; \boldsymbol{k}') \\ &= -\frac{1}{g(t)}\partial_t \int_{\Lambda_k,\Lambda_{k'}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot(\boldsymbol{x}-\boldsymbol{x}')} g(t)C(t, t'; \boldsymbol{k})C(t, t'; \boldsymbol{k}'),\end{aligned} \tag{A.1}$$

resulting in equation (2.25). Similarly,

$$\begin{aligned}\mathcal{C}_{\epsilon\epsilon}(t, t'; \boldsymbol{x}-\boldsymbol{x}') &= \frac{1}{2}\int_{\Lambda_k,\Lambda_{k'}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot(\boldsymbol{x}-\boldsymbol{x}')}(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})^2 C(t, t'; \boldsymbol{k})C(t, t'; \boldsymbol{k}') \\ &= \frac{1}{2g(t)}\partial_t^2 \int_{\Lambda_k,\Lambda_{k'}} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot(\boldsymbol{x}-\boldsymbol{x}')} g(t)C(t, t'; \boldsymbol{k})C(t, t'; \boldsymbol{k}')\end{aligned} \tag{A.2}$$

gives the equation (2.27).

When $\mathcal{H} \rightarrow \mathcal{H} - \sum_{t,\boldsymbol{x}} h_1(t, \boldsymbol{x})S(t, \boldsymbol{x})$, the solution to the corresponding Langevin equation is

$$\widehat{S}_1(t, \boldsymbol{k}; h_1) = \widehat{S}(t, \boldsymbol{k}) + \frac{\mathrm{e}^{-\omega_k t}}{\sqrt{g(t)}}\int_0^t \mathrm{d}\tau\, \mathrm{e}^{\omega_k \tau}\sqrt{g(\tau)}h_1(\tau, \boldsymbol{k}), \tag{A.3}$$

where the first term on the right-hand side is the unperturbed solution as given in equation (2.8). Differentiating the above expression with respect to $h_1(t', \boldsymbol{x}')$ and Fourier transforming back gives $\mathcal{R}_{11}(t, t'; \boldsymbol{x}-\boldsymbol{x}')$, while both $\mathcal{R}_{21}(t, t'; \boldsymbol{x}-\boldsymbol{x}')$ and $\mathcal{R}_{\epsilon 1}(t, t'; \boldsymbol{x}-\boldsymbol{x}')$ vanish since $\langle \widehat{S}(t, \boldsymbol{k})\rangle = 0$.

When $\mathcal{H} \rightarrow \mathcal{H} - \sum_{t,\boldsymbol{x}} h_2(t, \boldsymbol{x})S^2(t, \boldsymbol{x})$, the solution to the corresponding Langevin equation is

$$\widehat{S}_2(t, \boldsymbol{k}; h_2) = \widehat{S}(t, \boldsymbol{k}) + 2\int_0^t \mathrm{d}\tau\, \mathrm{e}^{\omega_k(\tau-t)}\sqrt{\frac{g(\tau)}{g(t)}}\int_{\Lambda_{k'}} \widehat{S}_2(\tau, \boldsymbol{k}'; h_2)h_2(\tau, \boldsymbol{k}-\boldsymbol{k}'). \tag{A.4}$$

Here $\widehat{S}_2(\tau, \boldsymbol{k}'; h_2)$ on the right-hand side can be replaced by $\widehat{S}(\tau, \boldsymbol{k}')$, while evaluating the response functions, since the difference is of order $O\left(h_2^2\right)$. Clearly, $\mathcal{R}_{12}(t, t'; \boldsymbol{x}-\boldsymbol{x}')$ vanishes since the initial magnetization is zero. Using the above equation we get

$$\left\langle \widehat{S}(t, \boldsymbol{k})\frac{\delta \widehat{S}_2(t, \boldsymbol{k}'; h_2)}{\delta h_2(t', \boldsymbol{x}')}\right\rangle_{h_2=0} = 2\sqrt{\frac{g(t')}{g(t)}}\,\mathrm{e}^{-\omega_{k'}(t-t')}C(t, t'; \boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}'}, \tag{A.5}$$

and then multiplying it by $2\exp(\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x})$ and integrating over $\Lambda_{\boldsymbol{k}}$ and $\Lambda_{\boldsymbol{k}'}$ results in equation (2.31). When we multiply equation (A.5) by $2\omega_{\boldsymbol{k}}\exp(\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x})$ and integrate over $\Lambda_{\boldsymbol{k}}$ and $\Lambda_{\boldsymbol{k}'}$, we get

$$\mathcal{R}_{\epsilon 2}(t,t';\boldsymbol{x}-\boldsymbol{x}') = 4\sqrt{\frac{g(t')}{g(t)}}C(t,t';\boldsymbol{x}-\boldsymbol{x}')\int_{\Lambda_{\boldsymbol{k}}}\omega_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\,\mathrm{e}^{-\omega_{\boldsymbol{k}}(t-t')}, \tag{A.6}$$

which is rewritten as in equation (2.32).

When $\mathcal{H} \to \mathcal{H} - \sum_{t,\boldsymbol{x}} h_\epsilon(t,\boldsymbol{x})\mathcal{O}_\epsilon(t,\boldsymbol{x})$, the solution to the corresponding Langevin equation $\widehat{S}_\epsilon(t,\boldsymbol{k};\epsilon) = \widehat{S}(t,\boldsymbol{k}) + \delta\widehat{S}_\epsilon(t,\boldsymbol{k};\epsilon)$, where

$$\delta\widehat{S}_\epsilon(t,\boldsymbol{k};\epsilon) = \int_0^t \mathrm{d}\tau\,\mathrm{e}^{\omega_{\boldsymbol{k}}(\tau-t)}\sqrt{\frac{g(\tau)}{g(t)}}\int_{\Lambda_{\boldsymbol{k}'}}(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})\widehat{S}_\epsilon(\tau,\boldsymbol{k}';\epsilon)h_\epsilon(\tau,\boldsymbol{k}-\boldsymbol{k}'). \tag{A.7}$$

From the above equation we get

$$\begin{aligned}\left\langle \widehat{S}(t,\boldsymbol{k})\frac{\delta\widehat{S}_\epsilon(t,\boldsymbol{k}';\epsilon)}{\delta h_\epsilon(t',\boldsymbol{x}')}\right\rangle_{\epsilon=0} &= \sqrt{\frac{g(t')}{g(t)}}\,\mathrm{e}^{\omega_{\boldsymbol{k}'}(t'-t)}\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}'}(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})C(t,t';\boldsymbol{k})\\ &= -\frac{\sqrt{g(t')}}{g(t)}\partial_t(\mathrm{e}^{-\omega_{\boldsymbol{k}'}(t-t')}\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}'}\sqrt{g(t)}C(t,t';\boldsymbol{k})).\end{aligned} \tag{A.8}$$

Now multiply this equation by $2\exp(\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x})$ and integrate over $\Lambda_{\boldsymbol{k}}$ and $\Lambda_{\boldsymbol{k}'}$ to get equation (2.34). If we multiply equation (A.8) by a factor $(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})$ then the only change in the last term is that ∂_t gets replaced by $-\partial_t^2$. Further, multiplying by $\exp(\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x})$ and integrating over $\Lambda_{\boldsymbol{k}}$ and $\Lambda_{\boldsymbol{k}'}$ results in equation (2.35).

When the temperature $T \to T + T'(t,\boldsymbol{x})$ is shifted then the solution $\widehat{S}_T(t,\boldsymbol{k};T')$ evolves just as given in equation (2.8), where the mean is $\langle\widehat{\eta}(t,\boldsymbol{k})\rangle_{T'} = 0$, but the variance becomes

$$\langle\widehat{\eta}(t,\boldsymbol{k})\widehat{\eta}(t',\boldsymbol{k}')\rangle_{T'} = \langle\widehat{\eta}(t,\boldsymbol{k})\widehat{\eta}(t',\boldsymbol{k}')\rangle + 2T'_{\boldsymbol{k}+\boldsymbol{k}'}(t)\delta(t-t'). \tag{A.9}$$

This implies $\langle\widehat{S}_T(t,\boldsymbol{k};T')\rangle$ is independent of T' and

$$\frac{\delta}{\delta T'(t',\boldsymbol{x}')}\langle\widehat{S}(t,\boldsymbol{k})\widehat{S}(t,\boldsymbol{k}')\rangle_{T'}\bigg|_{T'=0} = 2\frac{g(t')}{g(t)}\,\mathrm{e}^{-(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})(t-t')}\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}'}. \tag{A.10}$$

Now multiplying this equation by $\exp(\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x})$, and then integrating over $\Lambda_{\boldsymbol{k}}$ and $\Lambda_{\boldsymbol{k}'}$, results in equation (2.37). Similarly, multiplying by $\exp(\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x})$ along with the factor $(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})/2$, and then integrating over $\Lambda_{\boldsymbol{k}}$ and $\Lambda_{\boldsymbol{k}'}$, results in equations (2.38).

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